Nuclear dimension for an inclusion of unital C*-algebras

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Motivation

Question 1. Given an inclusion $P \subset A$ of unital C*-algebras, how closely related is P in A ?

If the Jones index of a II_1 factor N is finite with respect to a II_1-factor M, then M and N share many common properties such that as hyperfiniteness, property Γ, property T, as shown by [Pimsner and Popa:86].

In the case of inclusion of simple C*-algebras with a finite Watatani index we could not hope such an thing. In fact there are examples of inclusion $\mathrm{CAR} \subset \mathrm{CAR} \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ such that $\mathrm{CAR} \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ are not AF by [Blackadar:90] and [Elliott:89].

Let A be a unital C*-algebra and α an (amenable) action from a discrete group G on A, and $A \rtimes_\alpha G$ its crossed product algebra.

$$A \subset A \rtimes_\alpha G$$

Conditions for A	G	α	$A \rtimes_\alpha G$
(1) Simplicity	any	outer	◯
(2) Property (SP) + (1)	any	outer	◯
(3) Stable rank one	$\mathbf{Z}$	any	≤ 2
	finite	Rokhlin	◯
(2) + (3)	finite	any	≤ 2
(4) Real rank zero	finite	Rokhlin	◯
(5) The order on projections is determined by traces	finite	Rokhlin	◯
(6) AF, AI, AT, AD	finite	Rokhlin	◯
(7) AH with s.d.g. +(1)+(4)	finite	Rokhlin	◯
(8) $\mathcal{Z}$-stability	finite	Rokhlin	◯
	$\mathbf{Z}$	Rokhlin	◯

To generalize the previous results for an inclusion of unital C*-algebras $P \subset A$ we will give an attention to a canonical conditional expectation $E\colon A \rtimes_\alpha G \to A$ by $E(\sum_g a_g u_g) = a_0$, where $u\colon G \to A \rtimes_\alpha G$ is a unitary representation such that $u_g a u_g^* = \alpha_g(a)$ for any $a \in A$ and $g \in G$.

In this talk we assume that there is a (faithful) conditional expectation $E\colon A \to P$.

The following is the contents of this talk:

1. C*-index theory

2. Rokhlin property

3. Strongly self-absorbing

4. Nuclear dimension

C*-index theory

Definition 2. (Watatani:90)

Let $P \subset A$ be an inclusion of unital C*-algebras with a conditional expectation E from A onto P.

1. A *quasi-basis* for E is a finite set $\{(u_i, v_i)\}_{i=1}^{n} \subset A \times A$ such that for every $a \in A$,

$$a = \sum_{i=1}^{n} u_i E(v_i a) = \sum_{i=1}^{n} E(a u_i) v_i.$$

2. When $\{(u_i, v_i)\}_{i=1}^{n}$ is a quasi-basis for E, we define $\mathrm{Index} E$ by

$$\mathrm{Index} E = \sum_{i=1}^{n} u_i v_i.$$

   When there is no quasi-basis, we write $\mathrm{Index} E = \infty$. $\mathrm{Index} E$ is called the Watatani index of E.

Remark 3. We give several remarks about the above definitions.

1. $\mathrm{Index}E$ does not depend on the choice of the quasi-basis in the above formula, and it is a central element of A.

2. Once we know that there exists a quasi-basis, we can choose one of the form $\{(w_i, w_i^*)\}_{i=1}^m$, which shows that $\mathrm{Index}E$ is a positive element.

3. By the above statements, if A is a simple C^*-algebra, then $\mathrm{Index}E$ is a positive scalar.

4. If $\mathrm{Index}E < \infty$, then E is faithful, that is, $E(x^*x) = 0$ implies $x = 0$ for $x \in A$.

5. If $\mathrm{Index}E < \infty$, then there is a basic construction $C^*\langle A, e_p\rangle$ such that

$$C^*\langle A, e_p\rangle = \{\sum_{i=1}^{n} x_i e_P y_i : x_i, y_i \in A, n \in \mathbf{N}\}$$

and

$$P \subset A \subset C^*\langle A, e_p\rangle,$$

where e_p is called the Jones projection which satisfies $e_p a e_p = E(a)e_p$ for $a \in A$ and $e_p x = x e_p$ for $x \in P$.

6. If $\mathrm{Index}E$ is finite, then $\mathrm{Index}E$ is a central invertible element of A and there is the dual conditional expectation $\hat{E}$ from $C^*\langle A, e_P\rangle$ onto A such that

$$\hat{E}(xe_P y) = (\mathrm{Index}E)^{-1}xy \quad \text{for } x, y \in A$$

by Proposition 2.3.2 of [Watatani:90]. Moreover, $\hat{E}$ has a finite index and faithfulness.

The following is a model for an inclusion of unital C*-algebras:

Let A be a unital C*-algebra and α an action of a finite group G on A. Suppose that α is outer. Then

$$A^G \subset A \subset A \rtimes_\alpha G$$

is a basic construction.

Rokhlin property

Definition 4. (Izumi:04) Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rokhlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A' \cap A^\infty$ consisting of projections satisfying

$$(\alpha_g)_\infty(e_h) = e_{gh} \quad \text{for } g, h \in G.$$

We call $\{e_g\}_{g\in G}$ Rokhlin projections.

Here

$$\begin{aligned} c_0(A) &= \{(a_n) \in l^\infty(\mathbf{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\} \\ A^\infty &= l^\infty(\mathbf{N}, A)/c_0(A). \end{aligned}$$

We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences.

The following observation is our motivation to introduce the Rokhlin property for the inclusion of unital C*-algebras with a finite C*-index.

Proposition 5. (Kodaka-Osaka-Teruya 08) Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{\#G} \sum_{g \in G} \alpha_g(x) \quad \text{for } x \in A,$$

where $\#G$ is the order of G. Then α has the Rokhlin property if and only if there is a projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{\#G} \cdot 1$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

Definition 6. A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index}E)^{-1} \cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

The following is a key lemma to prove the main theorem

Lemma 7. (Kodaka-Osaka-Teruya:08)

Let $P \subset A$ be an inclusion of unital C*-algebras and E a conditional expectation from A onto P with a finite index. If E has the Rokhlin property with a Rokhlin projection $e \in A' \cap A^\infty$, then there is a unital linear map $\beta\colon A^\infty \to P^\infty$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty$.

In particular, $\beta_{|A}$ is a unital injective *-homomorphism and $\beta(x) = x$ for all $x \in P$.

Theorem 8. (Kodaka-Osaka-Teruya:09) Let a conditional expectation $E\colon A \to P$ be of index finite type and have the Rokhlin property.

1. If $\mathrm{tsr}(A) = 1$, then $\mathrm{tsr}(P) = 1$.

2. If $\mathrm{RR}(A) = 0$, then $\mathrm{RR}(P) = 0$.

Proof. We give the sketch of the proof of (1).

Let $x \in P$ and $\varepsilon > 0$. Since $\mathrm{tsr}(A) = 1$, there is an invertible $y \in A$ such that $\|x - y\| < \varepsilon$. Hence $\|\beta(x) - \beta(y)\| = \|x - \beta(y)\| < \varepsilon$. Since $\beta(y) = [(y_n)]$ is invertible in P^∞, we have an invertible $y_n \in P$ such that $\|x - y_n\| < \varepsilon$, and $\mathrm{tsr}(P) = 1$. □

Definition 9. Let A be a unital C*-algebra. We denote by $T(A)$ the set of all tracial states on A, equipped with the weak* topology. For any element of $T(A)$, we use the same letter for its standard extension to $M_n(A)$ for arbitrary n, and to $M_\infty(A) = \cup_{n=1}^{\infty} M_n(A)$.

We say that *the order on projections over a unital C*-algebra A is determined by traces* if whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \preceq q$.

Proposition 10. (Osaka-Teruya:10) Let $E\colon A \to P$ be of index finite type and have the Rokhlin property. Then the restriction map defines a bijection from the set $T(A)$ to the set $T(P)$.

Theorem 11. (O-Teruya:10) Let A be a unital C*-algebra such that the order on projections over A is determined by traces. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then the order on projections over P is determined by traces.

Strongly self-absorbing

A separable, unital C*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if it is infinite-dimensional and the map $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism $\varphi \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, that is, there is a suquence $(v_n)_{n \in \mathbf{N}}$ of unitaries in $\mathcal{D} \otimes \mathcal{D}$ satisfying

$$\|v_n^*(\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}(d))v_n - \varphi(d)\| \to 0 \ (n \to \infty) \ \forall d \in \mathcal{D}.$$

A C*-algebra A is called *$\mathcal{D}$-absorbing* if $A \otimes \mathcal{D} \cong A$.

Recall that separable unital C*-algebra $\mathcal{D}$ is said to have *approximately inner half flip* if the two natural inclusions of $\mathcal{D}$ into $\mathcal{D} \otimes \mathcal{D}$ as the first and second factor, respectively, are approximately unitarily equivalent, i.e., there is a sequence $(v_n)_{n \in \mathbf{N}}$ of unitaries in $\mathcal{D} \otimes \mathcal{D}$ such that

$$\|v_n(d \otimes 1_D)v_n^* - 1_D \otimes d\| \to 0 \ (n \to \infty)$$

for $d \in \mathcal{D}$.

In 1978 Effros and Rosenberg proved that if A is AF C*-algebra, A has approximate half inner flip if and only if A is a UHF algebra.

Note that if a separable unital C*-algebra A has approximately inner half-flip, then A is simple and nuclear.

Under the condition that separable unital C*-algebra $\mathcal{D}$ has approximately inner half flip, Toms and Winter gave the the characterization when $\mathcal{D}$ is strongly self-absorbing:

Theorem 12. (Toms-Winter:07) Let $\mathcal{D}$ be a separable unital C*-algebra such that $\mathcal{D}$ has an approximately inner half flip. Then $\mathcal{D}$ is strongly self-absorbing if and only if there are a unital *-homomorphism $\gamma\colon \mathcal{D}\otimes\mathcal{D} \to \mathcal{D}$ and an approximately central sequence of unital endmorphisms of $\mathcal{D}$

(i.e. $\exists(\varphi_n)\colon \mathcal{D} \to \mathcal{D}$ such that $\|[\varphi_n(d_1), d_2]\| \to 0$ $(n \to \infty)$ for $\forall d_1, d_2 \in \mathcal{D}$).

Using this characterization we show that if a conditional expectation $E\colon \mathcal{D} \to P$ for an inclusion of separable unital C*-algebras $P \subset \mathcal{D}$ with index finite, has the Rokhlin property and $\mathcal{D}$ is an inductive limit of *weakly semiprojective* C*-algebras and strongly self-absorbing, then P is strongly self-absorbing.

Note that known examples of strongly self-absorbing C*-algebras are UHF algebras of infinite type, the Jiang-Su algebras $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, and tensor products of $\mathcal{O}_\infty$ by UHF algebras of infinite type, that is, they belong to the class of inductive limits of weakly semiprojective C*-algebras.

Theorem 13. (Osaka-Teruya:11) Let $P \subset A$ be an inclusion of separable unital C*-algebras with index finite and A have approximately inner half flip. Suppose that E has the Rokhlin property and A is an inductive limit of weakly semiprojective C*-algebras. Then P has approximately inner half flip.

Theorem 14. (Osaka-Teruya:11) Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E\colon A \to P$ has the Rokhlin property and A is an inductive limit of weakly semiprojective C*-algebras and strongly self-absorbing. Then P is strongly self-absorbing.

Corollary 15. (Osaka-Teruya:11) Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E\colon A \to P$ has the Rokhlin property. Suppose that A is one of UHF-algebra of infinite type, $\mathcal{O}_2$, $\mathcal{O}_\infty$, and $\mathcal{O}_\infty \otimes$ UHF-algebra of infinite type. Then

1. $P \cong A$.

2. $C^*\langle A, e_P\rangle$ is stably isomorphic to A. If $A = \mathcal{O}_2$, then $C^*\langle A, e_P\rangle \cong \mathcal{O}_2$.

Nuclear dimension

Definition 16. (Winter-Zacharias:10)

Let A be a separable C*-algebra.

1. A completely positive map $\varphi\colon \bigoplus_{i=1}^{s} M_{r_i} \to A$ has order zero if it preserves orthogonality, i.e., $\varphi(e)\varphi(f) = \varphi(f)\varphi(e) = 0$ for $e, f \in \oplus_{i=1}^{s} M_{r_i}$ with $ef = fe = 0$.

2. A completely positive map $\varphi\colon \bigoplus_{i=1}^{s} M_{r_i} \to A$ is n-decomposable, there is a decomposition $\{1, \ldots, s\} = \coprod_{j=0}^{n} I_j$ such that the restriction of φ to $\oplus_{i\in I_j} M_{r_i}$ has ordere zero for each $j \in \{0, \ldots, n\}$.

3. A has decomposition rank n, $\mathrm{dr}A = n$, if n is the least integer such that the following holds : Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation property (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite dimensional F, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive contruction (= c. p. c) such that

   (a) $\|\varphi\psi(a_i) - a_i\| < \varepsilon$,
   (b) φ is n-decomposable.

   If no such n exists, we write $\mathrm{dr}A = \infty$.

4. A has nuclear dimension n, $\dim_{\mathrm{nuc}} A = n$, if n is the least integer such that the following holds : Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation property (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite dimensional F, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive such that

   (a) $\|\varphi\psi(a_i) - a_i\| < \varepsilon$
   (b) $\|\psi\| \leq 1$
   (c) φ is n-decomposable and each restriction $\varphi_{|\oplus_{i \in I_j} M_{r_i}}$ is c. p. c.

   If no such n exists, we write $\dim_{\mathrm{nuc}} A = \infty$.

The followings are basic facts about finite decomposition and nuclear dimension by [Kirchberg-Winter:04], [Winter:10], [Winter-Zacharias:10]:

1. If $\dim_{\mathrm{nuc}}(A) \leq n < \infty$, then A is nuclear.

2. For any C*-algebras $\dim_{\mathrm{nuc}} A \leq \mathrm{dr} A$.

3. $\dim_{\mathrm{nuc}} A = 0$ if and only if $\mathrm{dr} A = 0$ if and only if A is an AF algebra.

4. Nuclear dimension and decomposition rank in general do not coincide. Indeed, the Toeplitz algebra $\mathcal{T}$ has nuclear dimension at most 2, but its decomposition rank is infinity. Note that if $\mathrm{dr} A \leq n < \infty$, A is quasidiagonal, that is , stably finite. The Toeplitz algebra $\mathcal{T}$ has an isometry, and we know that $\mathcal{T}$ is infinite.

5. Let X be a locally compact Hausdorff space. Then

$$\dim_{\rm nuc} C_0(X) = {\rm dr} C_0(X).$$

In particular, if X is second countable,

$$\dim_{\rm nuc} C_0(X) = {\rm dr} C_0(X) = \dim X.$$

6. For any $n \in \mathbf{N}$ $\dim_{\rm nuc} A = \dim_{\rm nuc}(M_n(A)) = \dim_{\rm nuc}(A \otimes \mathcal{K})$.

7. If $B \subset A$ is full hereditary C*-algebra, then $\dim_{\rm nuc}(B) = \dim_{\rm nuc}(A)$.

Theorem 17. (Osaka-Teruya:10) Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a faithful conditional expectation of index finite. Suppose that E has the Rokhlin property.

1.
$$\mathrm{dr} P \leq \mathrm{dr} A$$

2.
$$\dim_{\mathrm{nuc}} P \leq \dim_{\mathrm{nuc}} A.$$

Corollary 18. Let A be a separable unital C*-algebra and α be an action of a finite group G on A. Suppose that α has the Rokhlin property. Then we have

1.

$$\mathrm{d}r(A^\alpha) \leq \mathrm{d}rA$$
$$\mathrm{dr}(A \rtimes_\alpha G) \leq \mathrm{d}rA.$$

2.

$$\dim_{\mathrm{nuc}}(A^\alpha) \leq \dim_{\mathrm{nuc}} A$$
$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \dim_{\mathrm{nuc}} A.$$

3. If A has locally finite nuclear dimension, then A^α and $A \rtimes_\alpha G$ have locally finite nuclear dimension.

Remark 19. When α does not have the Rokhlin property, generally the estimate in Corollary 18 would not be correct.

Indeed, let α be an symmetry action constructed by [Blackadar:90] such that $CAR^{\mathbf{Z}/2\mathbf{Z}}$ is not AF C*-algebra.

Then α does not have the Rokhlin property by [N. C. Phillips:06], and we know that $\dim_{\mathrm{nuc}}(CAR^{\mathbf{Z}/2\mathbf{Z}}) = 1$ (In fact, $CAR^{\mathbf{Z}/2\mathbf{Z}}$ can be realized as the inductive limits of $(C(S^1) \otimes M_{2^{2n-1}}) \oplus (C(S^1) \otimes M_{2^{2n-1}})$), but $\dim_{\mathrm{nuc}}(CAR) = 0$.

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